Int J Theor Phys (2010) 49: 2910–2917
DOI 10.1007/s10773-010-0482-3

Squeezed Wave Packets in Quantum Cosmology

Pouria Pedram

Received: 20 April 2010 / Accepted: 30 August 2010 / Published online: 7 September 2010

Abstract We use an appropriate initial condition for constructing squeezed wave packets in the context of Wheeler-DeWitt equation with complete classical description. This choice of initial condition does not alter the classical paths and only affect the quantum mechanical picture. To demonstrate the method, we consider an empty $4+1$-dimensional Kaluza-Klein quantum cosmology in the presence of a negative cosmological constant. We show that these wave packets do not disperse and sharply peak on the classical trajectories in the whole configuration space. So, the probability of finding the corresponding physical quantities approaches zero everywhere except on the classical paths.

Keywords Quantum cosmology · Classical-quantum correspondence · Kaluza-Klein cosmology

1 Introduction

The issue of multi-dimensional cosmology has attracted much attention and many papers has been appeared in the literature by numerous authors [1–3]. These efforts have begun by Kaluza and Klein who were interested to geometrically unify the gravitation and electromagnetism via introducing one undetectable extra dimension. This extra dimension is assumed periodic and its size can be made of the order of the Planck scale. However, in recent years, some theories have published in the literature which suggest the possibility of having large extra dimensions. Randall and Sundrum investigated a string theory inspired 5-dimensional Anti-de-Sitter bulk with an embedded 4-dimensional singular hypersurface which represents the universe [4, 5]. Arkani-Hamed *et al.* considered a multi-dimensional theory where the scale of gravity is taken to be the gauge unification scale rather than Planck mass in $4+d$-dimension [6, 7]. These investigations led to finding a reasonable explanation of the hierarchy problem (the huge disparity between fundamental constants in nature). Induced-Matter-Theory is an alternative theory which considers our 4-dimensional curved

P. Pedram (✉)
Plasma Physics Research Center, Science and Research Branch, Islamic Azad University, Tehran, Iran
e-mail: pouria.pedram@gmail.com

universe as an embedded subspace in a 5-dimensional vacuum manifold ($R_{AB} = 0$ where $A, B = 0, 1, 2, 3, 4$) [8]. Therefore, the presence of matter in our universe has only geometrical nature which comes from the induced vacuum solutions of the higher dimensions into our 4-dimensional hypersurface.

In quantum cosmology [10], one is usually concerned with constructing wave packets which are solutions of the underlying Wheeler-DeWitt (WDW) equation (see [9] and references therein). Most authors consider semi-classical approximations to the WDW equation and refer to regions in configuration space where these solutions are oscillatory or exponentially decaying as representing classically allowed or forbidden regions, respectively. These regions are mainly determined by the initial conditions for the wave function [9]. Some popular proposals for the initial conditions are the DeWitt boundary condition [10], the no-boundary proposal by Hartle and Hawking [11, 12], the tunneling condition by Vilenkin [13, 14], symmetric initial condition by Conardi and Zeh [15], and the Kiefer proposal [16–18]. These proposals are attractive to many authors because they lead to some classes of classical solutions represented by certain trajectories which possess important features such as predicting an inflationary phase. Also note that, the problem of the arrow of time or the origin of irreversibility in our universe is one of the most intriguing open problems which can be traced to the structure of the Wheeler-DeWitt equation [19].

Because of the hyperbolic nature of the WDW equation, we encounter with the problem of initial condition *i.e.* the freedom for choosing the initial wave function and its initial slope. On the other hand, in the classical domain, we have a unique classical trajectory which is usually singular. Some authors have used this freedom to construct wave packets which avoid singularities and represent the asymptotic classical behavior [20]. But, their solution is not unique and we can find uncountable singularity free solutions with the same asymptotic classical behavior. In fact, this result is not due to the quantum nature of the solutions and is purely related to the choice of initial conditions. In other words, it is possible to choose many initial conditions which result in non-vanishing wave packets at the singular points. So, the issue of classical and quantum correspondence is an important problem which is related to the selection of initial conditions. Now, we encounter with a significant question: Is it possible to choose initial conditions which results in the wave packets with *near-exact* classical behavior? This means that the amplitude of the wave packets should be almost zero beyond the classical trajectories. In [21–25], we have followed this path to construct wave packets which *approximately* correspond to their classical counterparts in various physical situations. In fact, we have proposed a prescription to make a connection between initial wave function and its initial slope. However, that prescription was not good enough to make the probability of finding the physical observables nearly zero beyond the classical paths.

Following the Bohmian interpretation of quantum mechanics the initial wave function and the initial slope of the wave function are related to the classical initial position and momentum, respectively [26]. In the case of the classical cosmology, after using the zero energy condition and fixing other variables, we obtain a unique classical trajectory. In fact, the zero energy condition connects the classical position and momentum variables. In the quantum domain, after the quantization procedure, this condition appears as WDW equation. Since the WDW equation is a second-order partial differential equation, it admits arbitrary initial wave functions and slopes. So, if we are interested to obtain the classical and quantum correspondence, we need to manually impose some constraints on their probability distributions or equivalently on the expansion coefficients.

In [21–25], we found that the classical and quantum correspondence fails for the small values of the classical radius. Because the parts of the wave packets interfere with each other which results in a non-classical scenario. Now, the question is that is this phenomena an intrinsic property of the quantum cosmology? In this paper, in order to answer this question, we use the advantages of the freedom in the selection of the wave function's initial condition. For instance, consider the following symmetric initial wave functions $\Psi_1(u,0) \equiv e^{-(u-u_0)^2} + e^{-(u+u_0)^2}$ and $\Psi_2(u,0) \equiv e^{-\alpha^2(u-u_0)^2} + e^{-\alpha^2(u+u_0)^2}$. Note that, both $\Psi_1(u,0)$ and $\Psi_2(u,0)$ have a common peak at $u = \pm u_0$. But, for the large values of α the later has a sharp peak at $u = \pm u_0$. So, if we relate the position of the peaks to the classical initial points, the second initial condition does not alter the classical picture. On the other hand, $\Psi_1(u,0)$ does not provide a near-exact classical description especially for $u_0 \lesssim 1$ (because the two Gaussians are not well-separated). Therefore, the second choice with $\alpha \gg 1$ solves the interference problem with suitable classical picture.

To demonstrate the method, we study the quantum cosmology aspects of an empty $4+1$-dimensional Kaluza-Klein model in the presence of a negative cosmological constant where the 4-dimensional part of the metric is Friedmann-Robertson-Walker (FRW) type. This model contains two scale factors, one is the usual FRW scale factor and another one is the scale factor of the extra dimension. At the classical level, after suitable change of variables, this model can be reduced to an exactly solvable oscillator-ghost-oscillator system which shows that the solutions are singular classically for the both scale factors. At the quantum level, after the quantization procedure, we need to solve the corresponding WDW equation. Although the quantum cosmology of this model is also studied in [27], but the issues of the initial condition and classical and quantum correspondence are not addressed therein. In fact, they freely put all except first few expansion coefficients equal to zero.

Note that, the mentioned initial condition in not so useful in ordinary quantum mechanics. Because, except the simple harmonic oscillator, for other potentials if we let α goes to infinity the wave function $\Psi(x,t)$ will loses its shape just after $t = 0$. Of course, for other values of α this result will also happens but in longer time. In fact, this is due to the parabolic nature of the Schrödinger equation which is first order in time variable $i\hbar\partial_t\psi = H\psi$. On the other hand, this scenario is very different in quantum cosmology. It is shown that for various form of the WDW equation (in addition to the model presented below), it is possible to find non-dispersed wave packets which peak on the classical paths [21–25]. Moreover, if we use the Bohmian interpretation, we can conclude that the wave packets preserve their shape for all time. So, this method has wide applications in quantum cosmology which is not limited to our model.

2 The model

Let us start with a $4+1$-dimensional metric in which its 4-dimensional part is assumed to be of FRW type

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] + a^2(t)d\rho^2, \tag{1}$$

where $k = +1, 0, -1$ correspond to a closed, flat or open universe and $R(t)$, $a(t)$ are the scale factors of the universe and extra dimension, respectively. Now, we can construct the Einstein-Hilbert action in the presence of the cosmological constant Λ

$$S = \int \sqrt{-g}(\mathcal{R} - \Lambda)dt\, d^3r\, d\rho, \tag{2}$$

where $g_{\mu\nu}$ is the metric, g is its determinant, and $\mathcal{R}$ is the Ricci scalar. The Ricci scalar corresponding to the above metric is $\mathcal{R}=6[\frac{\ddot{R}}{R}+\frac{k+\dot{R}^2}{R^2}]+2\frac{\ddot{a}}{a}+6\frac{\dot{R}}{R}\frac{\dot{a}}{a}$ where a dot represents differentiation with respect to time. Now, the effective Lagrangian can be obtained by integrating over spatial dimensions as

$$L=\frac{1}{2}R\dot{R}^2a+\frac{1}{2}R^2\dot{R}\dot{a}-\frac{1}{2}kRa+\frac{1}{6}\Lambda R^3a, \tag{3}$$

where R and a are the minisuperspace's variables. We can also simplify this Lagrangian by defining $\omega^2\equiv-\frac{2\Lambda}{3}$ and introducing new variables $u=\frac{1}{\sqrt{8}}[R^2+Ra-\frac{3k}{\Lambda}]$ and $v=\frac{1}{\sqrt{8}}[R^2-Ra-\frac{3k}{\Lambda}]$. In terms of new variables the Lagrangian takes the form $L=\frac{1}{2}[(\dot{u}^2-\omega^2u^2)-(\dot{v}^2-\omega^2v^2)]$ which results in the following Hamiltonian:

$$H=\frac{1}{2}\left[P_u^2+\omega^2u^2-P_v^2-\omega^2v^2\right], \tag{4}$$

where $P_u=\dot{u}$ and $P_v=\dot{v}$ are the canonical momenta conjugate to u and v, respectively. Above Hamiltonian represents an isotropic oscillator-ghost-oscillator system. Using the commutation relations $\{u,v\}=0$, $\{P_u,P_v\}=0$, $\{u,P_u\}=1$, $\{v,P_v\}=1$, and zero energy condition for the Hamiltonian $H=0$, we find the equations of motion for these variables, namely

$$u(t)=D\cos(\omega t-\theta_0),\qquad v(t)=D\sin(\omega t), \tag{5}$$

where D and θ_0 are constants which can be obtained after imposing the initial conditions. It is obvious that the above solutions represent Lissajous ellipsis which are singular in the present ($t=0$) and in the future ($t=\pi/\omega$).

3 Quantum cosmology

Now, let us study the quantum cosmology aspects of the model presented above. The Hamiltonian can be obtained upon quantization procedure $P_u\to-i\frac{\partial}{\partial u}$ and $P_v\to-i\frac{\partial}{\partial v}$. Therefore, one arrives at the WDW equation describing the corresponding quantum cosmology

$$H\Psi(u,v)=\left\{\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}-\omega^2u^2+\omega^2v^2\right\}\Psi(u,v)=0. \tag{6}$$

This equation is separable in the minisuperspace variables and the solutions can be written as

$$\Psi_n^\alpha(u,v)=\psi_n^\alpha(u)\psi_n^\alpha(v), \tag{7}$$

where $\psi_n^\alpha(z)=(\frac{\omega}{\pi})^{1/4}[\frac{H_n(\sqrt{\omega}\alpha z)}{\sqrt{2^n n!}}]e^{-\omega\alpha^2z^2/2}$. In the above expression, $H_n(z)$ is the Hermite polynomial and the orthonormality and completeness of the basis functions follow from those of the Hermite polynomials. The general form of the wave packets which satisfies the WDW equation (6) can be written as follows

$$\Psi^\alpha(u,v)=\sum_{n=\text{even}}A_n\psi_n^\alpha(u)\psi_n^\alpha(v)+i\sum_{n=\text{odd}}B_n\psi_n^\alpha(u)\psi_n^\alpha(v), \tag{8}$$

where the case $\alpha = 1$ corresponds to the solutions with approximate classical-quantum correspondence [21–25]. Since the potential in each direction is an even function, the eigenfunctions are separated in both even and odd categories. In fact, the coefficients A_n determine the initial wave function and the coefficients B_n determine the initial derivative of the wave function [21–25]. Since the underling WDW equation (6) is a second-order differential equation, A_n's and B_n's are arbitrary and independent variables. On the other hand, if we are interested to construct wave packets which simulate the classical behavior with known classical positions and momentums, all of these coefficients will not be independent. As we shall see, the usage of the real coefficients for the even terms and the imaginary coefficients for odd terms in the expansion, results in the symmetric solutions about v axis (or $\theta_0 = 0$ in (5)). Here, in order to study the general case ($\theta_0 \neq 0$), we need to choose appropriate complex expansion coefficients for both even and odd terms.

To study the initial conditions let us consider the general form of the wave packet near $v = 0$ (for details, see [21–25] which is also applicable for $\alpha \neq 1$). In this limit, the solution to the WDW equation is in the form of $\Psi_n^\alpha(u, v) = \psi_n^\alpha(u)\chi(v)$ where $\chi(v)$ is an oscillatory function of v. So, near $v = 0$ we have

$$\Psi_n^\alpha(u, v) = \sum_{n=\text{even}} a_n \cos(\sqrt{E_n} v)\psi_n^\alpha(u) + i \sum_{n=\text{odd}} b_n \sin(\sqrt{E_n} v)\psi_n^\alpha(u), \tag{9}$$

where E_n's are the separation constants and equal to the energy eigenvalues of the simple harmonic oscillator $E_n = (2n + 1)\omega$. Since we are interested to construct a wave packet with all classical properties, we need to assume a specific relationship between these coefficients. The prescription is that the coefficients have a same functional form *i.e.* $a_n = C(n, \alpha)$ for n even and $b_n = C(n, \alpha)$ for n odd. So, in terms of A_n and B_n we have $A_n = \frac{1}{\psi_n(0)} C(n, \alpha)$ for n even and $B_n = \frac{\sqrt{E_n}}{\psi_n'(0)} C(n, \alpha)$ for n odd. Note that, we need to specify $C(n, \alpha)$ in such a way that the initial wave function sharply peaks on the classical initial position. Now, using equations (8) and above relation for the expansion coefficients, we can explicitly write the form of the wave packet

$$\begin{aligned}\Psi^\alpha(u, v) = &\sum_{n=\text{even}} \frac{1}{H_n(0)} C(n, \alpha) \frac{H_n(\sqrt{\omega}\alpha u) H_n(\sqrt{\omega}\alpha v)}{(\pi/\omega)^{1/4}\sqrt{2^n n!}} e^{-\omega\alpha^2(u^2+v^2)/2} \\ &+ i \sum_{n=\text{odd}} \frac{\sqrt{2n+1}}{2n H_{n-1}(0)} C(n, \alpha) \frac{H_n(\sqrt{\omega}\alpha u) H_n(\sqrt{\omega}\alpha v)}{(\pi/\omega)^{1/4}\sqrt{2^n n!}} e^{-\omega\alpha^2(u^2+v^2)/2}.\end{aligned} \tag{10}$$

For instance consider the following form of the expansion coefficients:

$$C(n, \alpha) = \frac{\alpha^n \zeta^n e^{-\alpha^2|\zeta|^2/4}}{\sqrt{2^n n!}}, \tag{11}$$

where $\zeta = |\zeta| e^{-i\theta_0}$. The initial wave functions resulting from these coefficients contain two well-separated peaks centered at $\pm|\zeta|$. These two peaks correspond to classical initial ($t = 0$) and final ($t = \pi/\omega$) values of u, respectively. The squeezed wave packets can be obtained by $\alpha \to \infty$. At this limit, the initial wave function approaches zero everywhere except at the classical initial and final points which has a finite value. For the case $\alpha = 5$, $\theta_0 = \pi/4$ and $|\zeta| = 3$, we have depicted the resulting wave packet at the left part of Fig. 1. Moreover, as it can be seen from the right part of this figure, the larger values of α leads to narrower peaks in the initial wave function. Figure 2 shows that how the parameter α

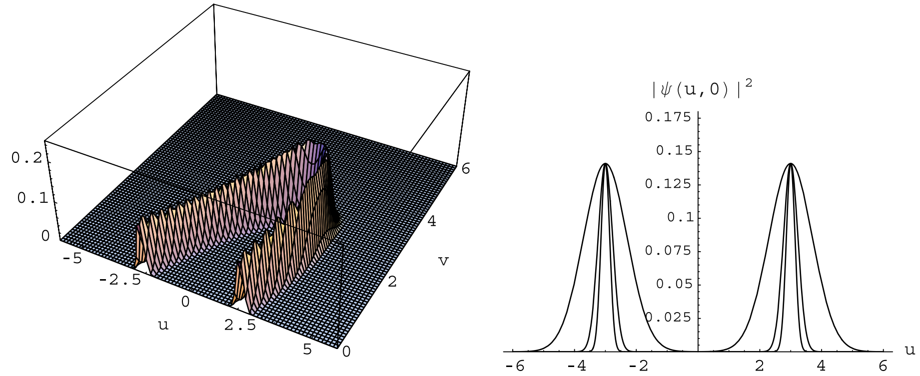

Fig. 1 *Left*, the square of the wave packet $|\psi(u, v)|^2$ for $\alpha = 5$, $\theta_0 = \pi/4$, $\omega = 1$, and $|\zeta| = 3$. *Right*, the square of the initial wave function for $\alpha = 1, 3, 5$, $\theta_0 = 0$ and $|\zeta| = 3$

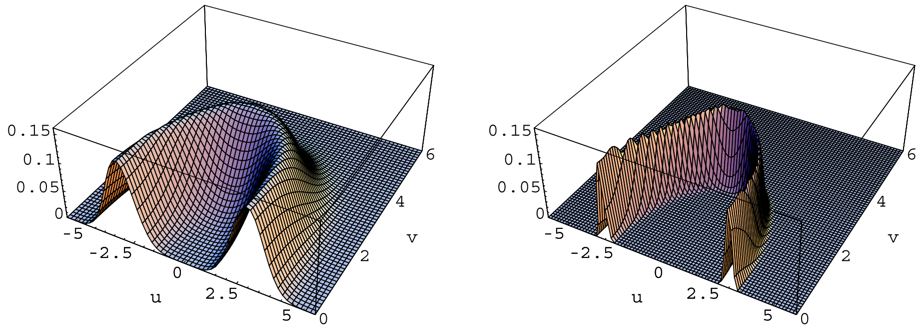

Fig. 2 The square of the wave packet $|\psi(u, v)|^2$ for $\alpha = 1$ (*left*) and $\alpha = 5$ (*right*). For both cases we set $\theta_0 = 0$, $\omega = 1$, and $|\zeta| = 3$

affects the form of the wave packets. So, the preference of larger values of α, from classical-quantum correspondence point of view, remains valid in the whole configuration space. Note that, we are free to choose any other appropriate initial condition. For instance, we can also choose $C'(n, \alpha) = n^k C(n, \alpha)$. For this case, the behavior of the resulting wave packets are similar to those which are shown in Figs. 1 and 2, but their initial peaks will not be centered at $\pm|\zeta|$ anymore.

Now, to obtain a good classical and quantum correspondence, we define the "classical" wave packet

$$\Psi_{cl}(u, v) \equiv \lim_{\alpha \to \infty} \Psi^{\alpha}(u, v), \tag{12}$$

which has the desired property $\Psi_{cl}(u, v) \neq 0$ at $\{u = u_{cl}, v = v_{cl}\}$ and $\Psi_{cl}(u, v) \simeq 0$ elsewhere. This wave packet exhibits other classical properties such as the behavior of the classical momentum throughout the trajectory. Note that, the height of the crest of the wave packets shows the constancy or periodicity of the classical momentum via (5). In fact, these wave packets, up to the WKB approximation, satisfy the classical probability relation on the

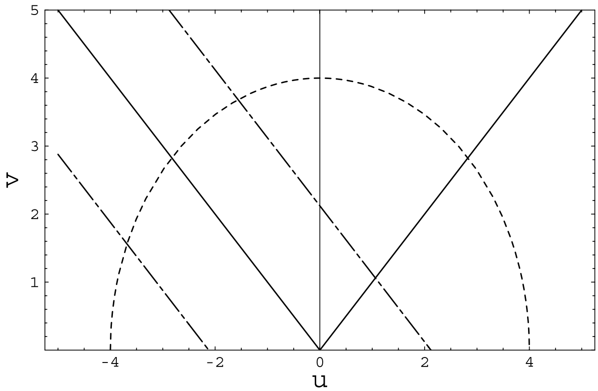

Fig. 3 The singularity borders (*solid lines* and *dot-dashed lines*) and the classical path (*dashed line*) for $\Lambda = -1$, $D = 4$, and $\theta_0 = 0$

classical path

$$|\Psi_{cl}|^2 \propto \frac{1}{P}, \tag{13}$$

where P is the classical momentum. In other words, the high probability density corresponds to low momentum and vice versa (Figs. 1 and 2). To make the correspondence between the classical and quantum mechanical results more concrete, we can also use the Bohmian interpretation of quantum mechanics [26]. In fact, the usage of above prescription for the expansion coefficients leads to the vanishing of quantum potential on the classical paths (since the "classical" wave packet exists only on the classical path, the quantum potential vanishes everywhere) and unification of classical and Bohmian trajectories. Also note that, a non-zero distribution of the initial slope of the wave packets is necessary to have a non-zero initial Bohmian momentum, although it is not compulsory from the mathematical point of view. One important property of these squeezed wave packets is that when we express them in terms of the original variables $\Psi(u, v) \to \Psi(R, a)$, the resulting wave packets are also squeezed states. However, for small values of α it is possible to have a wide wave packet in terms of $\{R, a\}$ variables which is correspond to a localized state in terms of $\{u, v\}$.

As we have stated before, this model is singular at the classical level. The singular points ($R = 0$ and $a = 0$) correspond to $v = -u - \frac{3}{\sqrt{2}}\frac{k}{\Lambda}$ and $v = u$, respectively. In Fig. 3, we have shown a classical trajectory (dashed line) and the singularity borders (solid lines and dot-dashed lines). The left solid line corresponds to $k = 0$ and dot-dashed lines correspond to $k = \pm 1$. The right solid line exists for all values of k. So, we have two singularities (for each k) during $t = 0$ and $t = \pi/\omega$. Since the wave packets do not vanish at these points, we have an indication that this model could be singular also at the quantum level.

4 Conclusions

We have presented an appropriate initial condition for the WDW equation which results in the squeezed wave packets with near-exact classical and quantum correspondence. We used this method to study an empty $4 + 1$-dimensional Kaluza-Klein quantum cosmology in the presence of a negative cosmological constant. We showed that for $\alpha \to \infty$ the wave functions vanish everywhere except on the classical paths and strongly peak on them. Moreover, the Bohmian interpretation of quantum mechanics and WKB approximation both predicted the correct corresponding classical properties. We finally showed that the model could be also singular at the quantum level.

Acknowledgement I would like to thank F. Darabi for useful discussions and valuable comments.

References

1. Bronnikov, K.A., Melnikov, V.N.: Ann. Phys. **239**, 40 (1995)
2. Kasper, U., Rainer, M., Zhuk, A.: Gen. Relativ. Gravit. **29**, 1123 (1997)
3. Mignemi, S., Schmidt, H.: J. Math. Phys. **39**, 998 (1998)
4. Randall, L., Sundrum, R.: Phys. Rev. Lett. **83**, 4690 (1999)
5. Randall, L., Sundrum, R.: Phys. Rev. Lett. **83**, 3370 (1999)
6. Arkani-Hamed, N., Dimopoulos, S., Kaloper, N., March-Russell, J.: Nucl. Phys. B **567**, 189 (2000)
7. Lyth, D.: Phys. Lett. B **448**, 191 (1999)
8. Wesson, P.S.: Space-Time-Matter: Modern Kaluza-Klein Theory. World Scientific, Singapore (1999)
9. Kiefer, C.: Quantum Gravity, 2nd edn. Oxford University Press, Oxford (2007)
10. DeWitt, B.S.: Phys. Rev. **160**, 1113 (1967)
11. Hawking, S.W.: Pontif. Acad. Sci. Scr. Varia **48**, 563 (1982)
12. Hartle, J.B., Hawking, S.W.: Phys. Rev. D **28**, 2960 (1983)
13. Vilenkin, A.: Phys. Rev. D **37**, 888 (1988)
14. Vilenkin, A.: In: Gibbons, G.W., Shellard, E.P.S., Rankin, S.J. (eds.) The Future of Theoretical Physics and Cosmology, pp. 649–666. Cambridge University Press, Cambridge (2003)
15. Conradi, H.D., Zeh, H.D.: Phys. Lett. A **154**, 321 (1991)
16. Kiefer, C.: Phys. Rev. D **38**, 1761 (1988)
17. Kiefer, C.: Phys. Lett. B **225**, 227 (1989)
18. Kiefer, C.: Nucl. Phys. B **341**, 273 (1990)
19. Zeh, H.D.: The Physical Basis of the Direction of Time, 4th edn. Springer, Berlin (2001)
20. Shojai, F., Molladavoudi, S.: Gen. Relativ. Gravit. **39**, 795 (2007)
21. Goushe, S.S., Sepangi, H.R., Pedram, P., Mirzaei, M.: Class. Quantum Gravity **24**, 4377 (2007)
22. Pedram, P., Jalalzadeh, S.: Phys. Lett. B **660**, 1 (2008). arXiv:0712.2593
23. Pedram, P.: Cosmol. J. Astropart. Phys. **07**, 006 (2008). arXiv:0806.1913
24. Pedram, P.: Phys. Lett. B **671**, 1 (2009). arXiv:0811.3668
25. Pedram, P.: Ann. Phys. (Berlin) **19**, 285 (2010)
26. Holland, P.R.: The Quantum Theory of Motion: An Account of the de Broglie-Bohm Interpretation of Quantum Mechanics. Cambridge University Press, Cambridge (1993)
27. Darabi, F., Sepangi, H.R.: Class. Quantum. Gravity **16**, 1565 (1999)